FI S15

1. Given that

$$2z^3 - 5z^2 + 7z - 6 \equiv (2z - 3)(z^2 + az + b)$$

where a and b are real constants,

(a) find the value of a and the value of b.

(2)

(b) Given that z is a complex number, find the three exact roots of the equation

$$2z^3 - 5z^2 + 7z - 6 = 0$$

(3)

a) $-5z^2 \equiv 2az^2 - 3z^2 \quad \therefore\ 2a - 3 = -5 \Rightarrow 2a = -2 \ \therefore\ \underline{a = -1}$

$-6 \equiv -3b \quad \therefore\ \underline{b = 2}$

b) $(2z-3)(z^2 - z + 2)$

$\hookrightarrow (z - \frac{1}{2})^2 - \frac{1}{4} + 2 = 0$

$(z - \frac{1}{2})^2 = -\frac{7}{4}$

$z - \frac{1}{2} = \pm \frac{1}{2}\sqrt{7}\,i$

$z = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}\,i$

$f(z) = 0 \Rightarrow z = \frac{3}{2},\ \frac{1}{2} + \frac{1}{2}\sqrt{7}\,i,\ \frac{1}{2} - \frac{1}{2}\sqrt{7}\,i$

2. Use the standard results for $\sum_{r=1}^{n} r$ and for $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} (3r-2)^2 = \frac{n}{2}\left(an^2 + bn + c\right)$$

where a, b and c are integers to be found.

(5)

$$\sum_{r=1}^{n} 9r^2 - 12r + 4 = 9\sum r^2 - 12\sum r + 4\sum 1$$

$$= \frac{9}{6}n(n+1)(2n+1) - \frac{12}{2}n(n+1) + 4n$$

$$= \frac{9}{6}n(n+1)(2n+1) - \frac{36}{6}n(n+1) + \frac{24}{6}n$$

$$= \frac{1}{6}n\left[18n^2 + 27n + 9 - 36n^2 - 36n + 24\right]$$

$$= \frac{1}{6}n\left[18n^2 - 9n - 3\right] = \frac{3}{6}n\left(6n^2 - 3n - 1\right)$$

$$= \frac{1}{2}n\left(6n^2 - 3n - 1\right)$$

3. It is given that α and β are roots of the equation

$$2x^2 - 7x + 4 = 0$$

(a) Find the exact value of $\alpha^2 + \beta^2$

(3)

(b) Find a quadratic equation which has roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$, giving your answer in the form $ax^2 + bx + c = 0$, where a, b and c are integers.

(3)

$x^2 - \frac{7}{2}x + 2 = 0$ $\quad$ $x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$\alpha + \beta = \frac{7}{2}$ $\quad$ $\alpha\beta = 2$ $\quad$ $(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$

$(\alpha+\beta)^2 = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$

$\frac{49}{4} = \alpha^2 + \beta^2 + 2(2)$ $\quad$ $\therefore \alpha^2 + \beta^2 = \frac{49}{4} - \frac{16}{4} = \frac{33}{4}$

b) $x^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)x + \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 0$

$x^2 - \left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)x + 1 = 0$

$x^2 - \frac{33}{8}x + 1 = 0$

4.

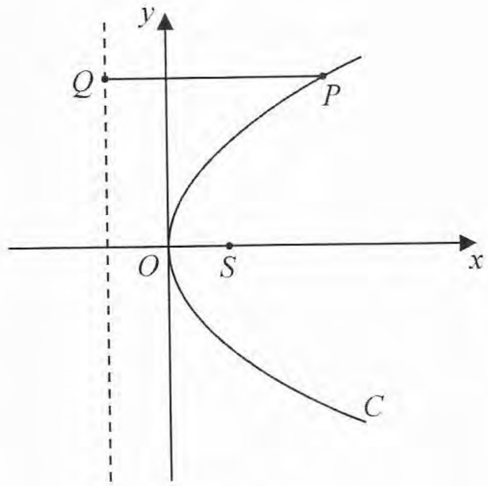


Figure 1

Figure 1 shows a sketch of the parabola C with equation $y^2 = 4ax$, where a is a positive constant. The point S is the focus of C and the point Q lies on the directrix of C. The point P lies on C where $y > 0$ and the line segment QP is parallel to the x-axis.

Given that the length of PS is 13

(a) write down the length of PQ.

(1)

Given that the point P has x coordinate 9

find

(b) the value of a,

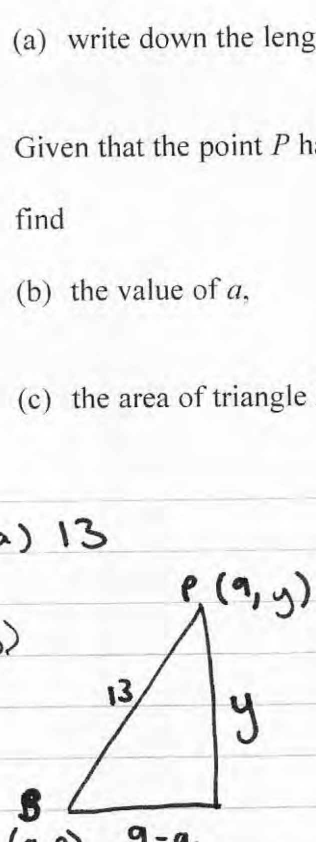
(2)

(c) the area of triangle PSQ.

(3)

a) 13

b)



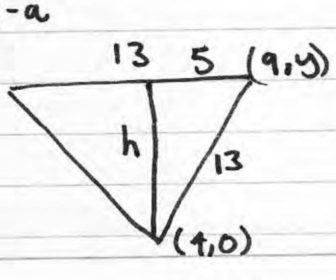
$= 4ax = 36a$

$(9-a)^2 + y^2 = 13^2 \Rightarrow 81 - 18a + a^2 + 36a = 169$

$a^2 + 18a - 88 = 0$ $\quad (a+22)(a-4) = 0$

$\therefore \underline{a = 4}$

c)



$h^2 = 13^2 - 5^2$ $\quad h^2 = 144 \therefore h = 12$

Area $= \frac{1}{2}bh = \frac{1}{2}(13)\cancel{5}_{12} = \cancel{\frac{65}{2}}$

$= 78$

5. In the interval $2 < x < 3$, the equation

$$6 - x^2 \cos\left(\frac{x}{5}\right) = 0, \text{ where } x \text{ is measured in radians}$$

has exactly one root α.

(a) Starting with the interval [2, 3], use interval bisection twice to find an interval of width 0.25 which contains α.

(4)

(b) Use linear interpolation once on the interval [2, 3] to find an approximation to α. Give your answer to 2 decimal places.

(3)

a)

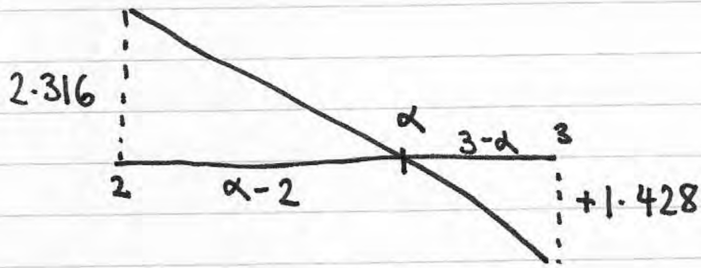
x	2·00	2·25	2·50	2·75	3·00
$f(x)$	2·3		0·5	−0·4	−1·4
	>0		>0	<0	<0

$f(x) = 6 - x^2\cos(\frac{1}{5}x)$

root

$\alpha \in (2.50, 2.75)$

b)



$$\frac{3-\alpha}{1.428} = \frac{\alpha-2}{2.316}$$

$$6.947 - 2.316\alpha = 1.428\alpha - 2.856$$

$$9.803 = 3.744\alpha$$

$$\alpha = 2.62$$

6. The rectangular hyperbola, H, has cartesian equation

$$xy = 36$$

The three points $P\left(6p, \frac{6}{p}\right)$, $Q\left(6q, \frac{6}{q}\right)$ and $R\left(6r, \frac{6}{r}\right)$, where p, q and r are distinct, non-zero values, lie on the hyperbola H.

(a) Show that an equation of the line PQ is

$$pqy + x = 6(p + q)$$

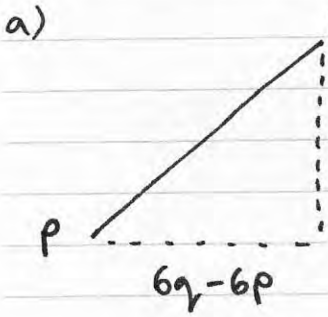
(4)

Given that PR is perpendicular to QR,

(b) show that the normal to the curve H at the point R is parallel to the line PQ.

(6)

a)



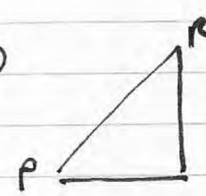
$\frac{6}{q} - \frac{6}{p} = \frac{6(p-q)}{pq}$ $\quad m_{PQ} = \frac{\frac{\cancel{6}(p-q)}{pq}}{\cancel{6}(q-p)} = \frac{-1}{pq}$

$y - \frac{6}{p} = \frac{-1}{pq}(x - 6p) \Rightarrow pqy - 6q = 6p - x$

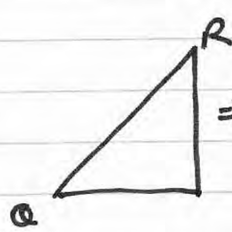
$pqy + x = 6p + 6q$

$\therefore pqy + x = 6(p+q)$ #

b)



$\Rightarrow M_{PR} = \frac{-1}{pr}$



$\Rightarrow M_{QR} = \frac{-1}{qr}$

So perp $= qr$

$\therefore -\frac{1}{pr} = qr \Rightarrow r^2 = \frac{-1}{pq}$

$y = \frac{36}{x} = 36x^{-1} \Rightarrow \frac{dy}{dx} = -36x^{-2} = \frac{-36}{x^2}$ $\quad$ at R $x = 6r$

m_t at H $= \frac{-36}{36r^2} = \frac{-1}{r^2}$ $\quad M_n$ at $H = r^2$ $\quad r^2 = -\frac{1}{pq} =$ gradient PQ

#

7.

$$z = -3k - 2k\mathrm{i}, \text{ where } k \text{ is a real, positive constant.}$$

(a) Find the modulus and the argument of z, giving the argument in radians to 2 decimal places and giving the modulus as an exact answer in terms of k.

(3)

(b) Express in the form $a + \mathrm{i}b$, where a and b are real and are given in terms of k where necessary,

(i) $\dfrac{4}{z + 3k}$

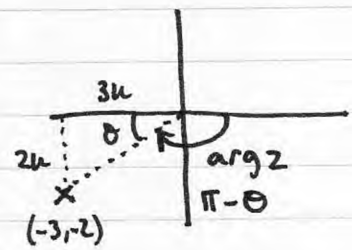
(ii) z^2

(5)

(c) Given that $k = 1$, plot the points A, B, C and D representing z, z^*, $\dfrac{4}{z+3k}$ and z^2 respectively on a single Argand diagram.

(3)

a) $|z| = \sqrt{(3k)^2 + (2k)^2} = \sqrt{13k^2} = k\sqrt{13}$



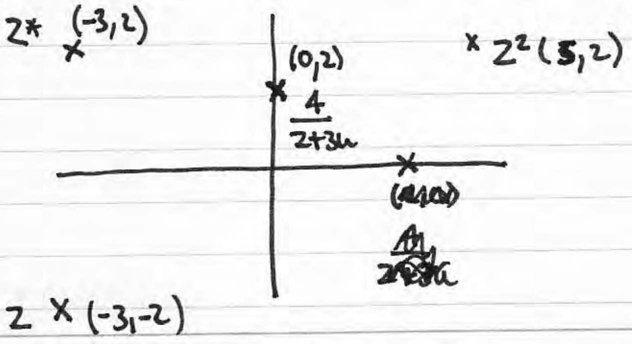
$\theta = \tan^{-1}\left(\frac{2k}{3k}\right) = 0.588$

$\therefore \arg(z) = -(\pi - 0.588) = -2.55^c$

b) i) $\dfrac{4}{z+3k} = \dfrac{4}{-2k\mathrm{i}} \times \dfrac{2k\mathrm{i}}{2k\mathrm{i}} = \dfrac{8k\mathrm{i}}{4k^2} = \dfrac{2}{k}\mathrm{i}$

ii) $z^2 = (-3k-2k\mathrm{i})(-3k-2k\mathrm{i}) = 9k^2 - 4k^2 + 12k^2\mathrm{i} = 5k^2 + 2k^2\mathrm{i}$

c) $k=1 \Rightarrow z = -3-2\mathrm{i} \quad z^* = -3+2\mathrm{i} \quad \dfrac{4}{z+3k} = 2\mathrm{i} \quad z^2 = 5+2\mathrm{i}$



8.

$$\mathbf{P} = \begin{pmatrix} 3a & -4a \\ 4a & 3a \end{pmatrix}, \text{ where } a \text{ is a constant and } a > 0$$

(a) Find the matrix $\mathbf{P}^{-1}$ in terms of a.

(3)

The matrix $\mathbf{P}$ represents the transformation U which transforms a triangle T_1 onto the triangle T_2.
The triangle T_2 has vertices at the points $(-3a, -4a)$, $(6a, 8a)$, and $(-20a, 15a)$.

(b) Find the coordinates of the vertices of T_1

(3)

(c) Hence, or otherwise, find the area of triangle T_2 in terms of a.

(3)

The transformation V, represented by the 2×2 matrix $\mathbf{Q}$, is a rotation through an angle α **clockwise** about the origin, where $\tan \alpha = \frac{4}{3}$ and $0 < \alpha < \frac{\pi}{2}$

(d) Write down the matrix $\mathbf{Q}$, giving each element as an exact value.

(2)

The transformation U followed by the transformation V is the transformation W. The matrix $\mathbf{R}$ represents the transformation W.

(e) Find the matrix $\mathbf{R}$.

(2)

$$P^{-1} = \frac{1}{\det P}\begin{pmatrix} 3a & 4a \\ -4a & 3a \end{pmatrix} \qquad P^{-1} = \frac{+1}{25a^2}\begin{pmatrix} 3a & 4a \\ -4a & 3a \end{pmatrix}$$

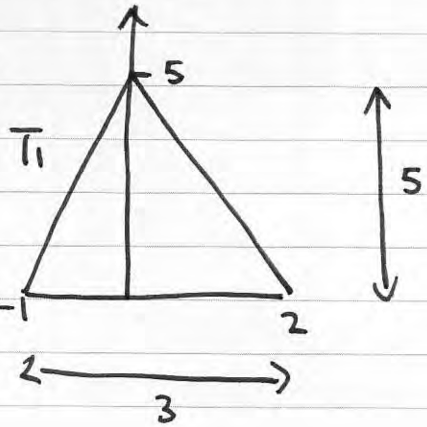
$$\det P = +16a^2 + 9a^2$$
$$= +25a^2$$

$$T_1 = \frac{+1}{25a^2}\begin{pmatrix} 3a & 4a \\ -4a & 3a \end{pmatrix}\begin{pmatrix} -3a & 6a & -20a \\ -4a & 8a & 15a \end{pmatrix}$$

$$= \frac{+1}{25a^2}\begin{pmatrix} -25a^2 & 50a^2 & 0 \\ 0 & 0 & 125a^2 \end{pmatrix} = \begin{pmatrix} -1 & +2 & 0 \\ 0 & 0 & +5 \end{pmatrix}$$

$T_1 \quad (-1, 0) \, ; (+2, 0) \, ; (0, +5)$

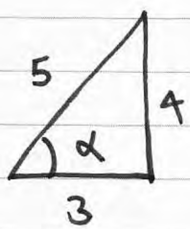
c)



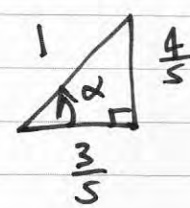
$\text{Area } T_1 = \frac{15}{2}$

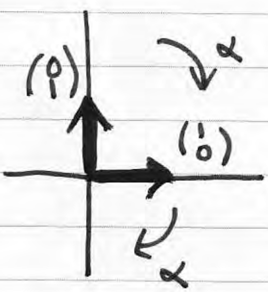
$\text{Area } T_2 = \frac{15}{2} \times 25a^2 = \frac{375}{2}a^2$

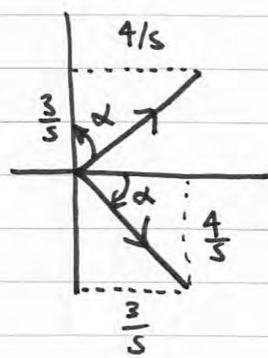
d)



$\tan\alpha = \frac{4}{3}$ $\Rightarrow$







$Q \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$

$Q = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$

e) $R = QP = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3a & -4a \\ 4a & 3a \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 25a & 0 \\ 0 & 25a \end{pmatrix}$

$\therefore R = \begin{pmatrix} 5a & 0 \\ 0 & 5a \end{pmatrix}$

9. (i) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} r^2(2r-1) = \frac{1}{6}n(n+1)(3n^2+n-1)$$

(6)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^n = \begin{pmatrix} 6n+1 & -12n \\ 3n & 1-6n \end{pmatrix}$$

(6)

$n=1$ LHS $= 1^2(2\times1-1) = 1$ RHS $= \frac{1}{6}(1)(1+1)(3+1-1)$

$= \frac{1}{6} \times 2 \times 3 = 1$

$\therefore$ LHS = RHS $\therefore$ true for $n=1$

assume true for $n=k$ $\therefore$ $\sum_1^k r^2(2r-1) = \frac{1}{6}k(k+1)(3k^2+k-1)$

$n=k+1$ $\sum_1^{k+1} r^2(2r-1) = \sum_1^k r^2(2r-1) + (k+1)^2(2k+2-1)$

$= \frac{1}{6}k(k+1)(3k^2+k-1) + \frac{6}{6}(k+1)^2(2k+1)$

$= \frac{1}{6}(k+1)\left[3k^3+k^2-k+6(2k^2+3k+1)\right]$

$= \frac{1}{6}(k+1)\left[3k^3+13k^2+17k+6\right]$ = LHS

RHS $= \frac{1}{6}(k+1)(k+1+1)(3(k+1)^2+(k+1)-1)$

$= \frac{1}{6}(k+1)(k+2)(3k^2+6k+3+k)$

$= \frac{1}{6}(k+1)(k+2)(3k^2+7k+3)$ $\therefore$ LHS = RHS

$= \frac{1}{6}(k+1)(3k^3+13k+17k+6)$ # $\therefore$ true for $n=k+1$

true for $n=1$, true for $n=k+1$ if true for $n=k$

$\therefore$ by Mathematical induction true for all $n \in \mathbb{Z}^+$

$n=1$ $\quad LHS = \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^1 = \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}$

$\therefore LHS = RHS$

$RHS = \begin{pmatrix} 6(1)+1 & -12(1) \\ 3(1) & 1-6(1) \end{pmatrix} = \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}$

$\therefore$ true for $n=k$.

assume true for $n=k$ $\quad \therefore \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^k = \begin{pmatrix} 6k+1 & -12k \\ 3k & 1-6k \end{pmatrix}$

$n=k+1$

$LHS = \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^{k+1} = \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^k$

$= \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} 6k+1 & -12k \\ 3k & 1-6k \end{pmatrix} = \begin{pmatrix} 42k+7-36k & -84k-12+72k \\ 18k+3-15k & -36k-5+30k \end{pmatrix}$

$LHS = \begin{pmatrix} 6k+7 & -12k-12 \\ 3k+3 & -6k-5 \end{pmatrix}$

$RHS = \begin{pmatrix} 6(k+1)+1 & -12(k+1) \\ 3(k+1) & 1-6(k+1) \end{pmatrix} = \begin{pmatrix} 6k+7 & -12k-12 \\ 3k+3 & -6k-5 \end{pmatrix}$

$\therefore LHS = RHS$ $\quad \therefore$ true for $n=k+1$

$\therefore$ true for $n=1$, true for $n=k+1$ if true for $n=k$

$\therefore$ by Mathematical Induction true for all $n \in \mathbb{Z}^+$